

On the r -matrix structure of the hyperbolic BC_n Sutherland model

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Abstract

Working in a symplectic reduction framework, we construct a dynamical r -matrix for the classical hyperbolic BC_n Sutherland model with *three independent coupling constants*. We also examine the Lax representation of the dynamics and its equivalence with the Hamiltonian equation of motion.

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1 Introduction

The Calogero–Moser–Sutherland-type many-particle models are intensively studied integrable systems with deep connections to many important branches of mathematics and physics. As a classical Hamiltonian system, the hyperbolic BC_n Sutherland model is defined on the cotangent bundle of the open subset

$$\mathfrak{c} = \{q = (q_1, \dots, q_n) \in \mathbb{R}^n \mid q_1 > \dots > q_n > 0\} \subset \mathbb{R}^n. \quad (1.1)$$

For convenience we identify the phase space $T^*\mathfrak{c}$ with the product manifold

$$\mathcal{P}^S = \mathfrak{c} \times \mathbb{R}^n = \{(q, p) \mid q \in \mathfrak{c}, p \in \mathbb{R}^n\}, \quad (1.2)$$

endowed with the standard symplectic form

$$\omega^S = \sum_{c=1}^n dq_c \wedge dp_c. \quad (1.3)$$

The dynamics is governed by the interacting many-body Hamiltonian

$$\begin{aligned} H^S = & \frac{1}{2} \sum_{c=1}^n p_c^2 + \sum_{c=1}^n \left(\frac{g_1^2}{\sinh^2(q_c)} + \frac{g_2^2}{\sinh^2(2q_c)} \right) \\ & + \sum_{1 \leq a < b \leq n} \left(\frac{g^2}{\sinh^2(q_a - q_b)} + \frac{g^2}{\sinh^2(q_a + q_b)} \right) \end{aligned} \quad (1.4)$$

with coupling constants $g^2, g_1^2, g_2^2 \in \mathbb{R}$ satisfying $g^2 > 0$, $g_2^2 \geq 0$ and $g_1^2 > -\frac{1}{4}g_2^2$.

By applying the projection method on the geodesic system of the non-compact Riemannian symmetric space $SU(n+1, n)/S(U(n+1) \times U(n))$, Olshanetsky and Perelomov constructed a Lax representation of the BC_n Sutherland dynamics and analyzed the issue of solvability as well, but only under the restrictive assumption $g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0$ (for details see e.g. [1], [2], [3]). As the algebraic methods prevailed, the Lax representation of the dynamics was soon established for arbitrary values of the coupling constants (see e.g. [4], [5], [6], [7]). Somewhat surprisingly, the symplectic reduction derivation of the BC_n Sutherland model with three *independent* coupling constants is only a relatively recent development [8]. Besides providing a nice geometric picture and an efficient solution algorithm, the symplectic reduction approach has also allowed us to construct action-angle variables to the BC_n Sutherland model and to establish its duality with the rational BC_n Ruijsenaars–Schneider–van Diejen system (see [9]). Sticking to the powerful machinery of symplectic reduction, in this paper we construct a dynamical r -matrix for the BC_n Sutherland model. By accomplishing this task we generalize the results of Avan, Babelon and Talon on the r -matrix structure of the hyperbolic C_n Sutherland model, which appeared in their paper [10].

The rest of the paper is organized as follows. In order to keep the presentation self-contained, in the next section we provide a brief account on the group theoretic and symplectic geometric background underlying the symplectic reduction derivation of the hyperbolic BC_n Sutherland model. Built upon the reduction approach outlined in Section 2, in Section 3 we construct a q -dependent dynamical r -matrix for the most general hyperbolic BC_n Sutherland model with three independent coupling constants. The new results are summarized concisely in Theorems 4 and 5. Subsequently, in Section 4, we offer a short discussion on possible applications and related open problems. Finally, some auxiliary material on the Lie algebra $\mathfrak{u}(n, n)$ can be found in an appendix.

2 Preliminaries

In this section we review the symplectic reduction derivation of the hyperbolic BC_n Sutherland model. For convenience, we closely follow the ideas and conventions presented in [9].

2.1 Group theoretic background

Take an arbitrary positive integer $n \in \mathbb{N}$, let $N = 2n$, and consider the $N \times N$ matrix

$$C = \begin{bmatrix} 0_n & \mathbf{1}_n \\ \mathbf{1}_n & 0_n \end{bmatrix}. \quad (2.1)$$

The matrix Lie group

$$G = \{y \in GL(N, \mathbb{C}) \mid y^* C y = C\} \quad (2.2)$$

provides an appropriate model of the real reductive Lie group $U(n, n)$. Its Lie algebra

$$\mathfrak{g} = \mathfrak{u}(u, n) = \{Y \in \mathfrak{gl}(N, \mathbb{C}) \mid Y^* C + C Y = 0\} \quad (2.3)$$

comes naturally equipped with the Ad-invariant symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}, \quad (Y, \tilde{Y}) \mapsto \langle Y, \tilde{Y} \rangle = \text{tr}(Y \tilde{Y}). \quad (2.4)$$

The fixed-point set of the Cartan involution $\Theta(y) = (y^{-1})^*$ ($y \in G$) can be identified as

$$K = \{y \in G \mid \Theta(y) = y\} \cong U(n) \times U(n), \quad (2.5)$$

meanwhile the corresponding Lie algebra involution $\theta(Y) = -Y^*$ ($Y \in \mathfrak{g}$) naturally induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the eigenspaces

$$\mathfrak{k} = \ker(\theta - \text{Id}_{\mathfrak{g}}) \quad \text{and} \quad \mathfrak{p} = \ker(\theta + \text{Id}_{\mathfrak{g}}). \quad (2.6)$$

That is, each $Y \in \mathfrak{g}$ can be uniquely decomposed as $Y = Y_+ + Y_-$ with $Y_+ \in \mathfrak{k}$ and $Y_- \in \mathfrak{p}$. Note that the bilinear form (2.4) is negative definite on the subalgebra \mathfrak{k} , whereas it is positive definite on the complementary subspace \mathfrak{p} .

Now, with each real n -tuple $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ we associate the $N \times N$ diagonal matrix

$$Q = \text{diag}(q_1, \dots, q_n, -q_1, \dots, -q_n) \in \mathfrak{p}. \quad (2.7)$$

Clearly the subset $\mathfrak{a} = \{Q \in \mathfrak{p} \mid q \in \mathbb{R}^n\}$ is a maximal Abelian subspace in \mathfrak{p} , which can be naturally identified with \mathbb{R}^n . Under the adjoint action, the centralizer of \mathfrak{a} in K is the subgroup

$$M = Z_K(\mathfrak{a}) = \{\text{diag}(e^{i\chi_1}, \dots, e^{i\chi_n}, e^{i\chi_1}, \dots, e^{i\chi_n}) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\} \subset K \quad (2.8)$$

with Lie algebra

$$\mathfrak{m} = \{\text{diag}(i\chi_1, \dots, i\chi_n, i\chi_1, \dots, i\chi_n) \mid \chi_1, \dots, \chi_n \in \mathbb{R}\} \subset \mathfrak{k}. \quad (2.9)$$

Let \mathfrak{a}^\perp (respectively \mathfrak{m}^\perp) denote the set of the off-diagonal elements of \mathfrak{p} (respectively \mathfrak{k}); then with respect to the bilinear form (2.4) we have the refined *orthogonal* decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp \oplus \mathfrak{a} \oplus \mathfrak{a}^\perp. \quad (2.10)$$

Practically, each Lie algebra element $Y \in \mathfrak{g}$ can be decomposed as

$$Y = Y_{\mathfrak{m}} + Y_{\mathfrak{m}^\perp} + Y_{\mathfrak{a}} + Y_{\mathfrak{a}^\perp} \quad (2.11)$$

with unique components belonging to the subspaces indicated by the subscripts.

Notice that for each $q \in \mathbb{R}^n$ the operator $\text{ad}_Q = [Q, \cdot] \in \mathfrak{gl}(\mathfrak{g})$ leaves the subspace $\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp$ invariant; therefore the restricted operator

$$\widetilde{\text{ad}}_Q = \text{ad}_Q|_{\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp} \in \mathfrak{gl}(\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp) \quad (2.12)$$

is well defined. Recall that the regular part of \mathfrak{a} consists of those diagonal matrices Q , for which the linear operator $\widetilde{\text{ad}}_Q$ is invertible. Clearly the standard Weyl chamber

$$\{Q \in \mathfrak{a} \mid q_1 > \dots > q_n > 0\} \quad (2.13)$$

is a connected component of the regular part of \mathfrak{a} . For simplicity, in the rest of the paper we shall identify this Weyl chamber with the configuration space \mathfrak{c} (1.1).

In deriving the Sutherland model from symplectic reduction, the so-called KAK decomposition of G plays a crucial role. It basically says that the map

$$\mathbb{R}^n \times K \times K \ni (q, k_L, k_R) \mapsto k_L e^Q k_R^{-1} \in G \quad (2.14)$$

is *onto*. Let G_{reg} denote the image of $\mathfrak{c} \times K \times K$ under the above map. As is known, the subset of regular elements, G_{reg} , is an open and dense submanifold of G . Moreover, the smooth map

$$\pi: \mathfrak{c} \times K \times K \rightarrow G_{\text{reg}}, \quad (q, k_L, k_R) \mapsto k_L e^Q k_R^{-1} \quad (2.15)$$

is a principal M -bundle in a natural manner. Consequently, we arrive at the natural identification $G_{\text{reg}} \cong \mathfrak{c} \times (K \times K)/M_*$, where M_* stands for the diagonal embedding of M (2.8) into the product Lie group $K \times K$. That is, M_* consists of the pairs $(m, m) \in K \times K$ with $m \in M$.

We conclude this subsection with a brief excursion on certain adjoint orbit of \mathfrak{k} , which is at the heart of the symplectic reduction derivation of the BC_n Sutherland model. For this, let us consider the following set of column vectors

$$S = \{V \in \mathbb{C}^N \mid CV + V = 0, V^*V = N\}, \quad (2.16)$$

which can be seen as a sphere of real dimension $2n - 1$. With each vector $V \in S$ we associate the matrix

$$\xi(V) = i\mu(VV^* - \mathbf{1}_N) + i(\mu - \nu)C \in \mathfrak{k}, \quad (2.17)$$

where $\mu, \nu \in \mathbb{R} \setminus \{0\}$ are arbitrary non-zero real parameters. Let us now introduce the distinguished column vector $E \in S$ with components

$$E_a = -E_{n+a} = 1 \quad (a \in \mathbb{N}_n = \{1, \dots, n\}). \quad (2.18)$$

Notice that the adjoint orbit in \mathfrak{k} passing through the element $\xi(E)$ has the form

$$\mathcal{O} = \mathcal{O}(\xi(E)) = \{\xi(V) \in \mathfrak{k} \mid V \in S\}. \quad (2.19)$$

More precisely, with the free action $U(1) \times S \ni (e^{i\psi}, V) \mapsto e^{i\psi}V \in S$, the map

$$\xi: S \rightarrow \mathcal{O}, \quad V \mapsto \xi(V) \quad (2.20)$$

is a principal $U(1)$ -bundle. Therefore the identification $\mathcal{O} \cong S/U(1)$ is immediate.

2.2 The Sutherland model from symplectic reduction

We continue with a short review on the symplectic structure of the cotangent bundle of G . For convenience, we trivialize this bundle by left translations. Therefore, making use of the linear isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ induced by (2.4), we can think of the product manifold $\mathcal{P} = G \times \mathfrak{g}$ as an appropriate model of T^*G . At each point $(y, Y) \in \mathcal{P}$ the canonical symplectic form $\omega \in \Omega^2(\mathcal{P})$ can be written as

$$\omega_{(y,Y)}(\Delta y \oplus \Delta Y, \delta y \oplus \delta Y) = \langle y^{-1} \Delta y, \delta Y \rangle - \langle y^{-1} \delta y, \Delta Y \rangle + \langle [y^{-1} \Delta y, y^{-1} \delta y], Y \rangle, \quad (2.21)$$

where $\Delta y \oplus \Delta Y$ and $\delta y \oplus \delta Y$ are arbitrary elements belonging to the tangent space $T_y G \oplus \mathfrak{g} \cong T_{(y,Y)} \mathcal{P}$. Turning to the adjoint orbit \mathcal{O} (2.19), remember that it also carries a natural symplectic structure induced by the Kirillov–Kostant–Souriau symplectic form $\omega^\mathcal{O} \in \Omega^2(\mathcal{O})$. Let us keep in mind that at each point $\rho \in \mathcal{O}$ it takes the form

$$\omega_\rho^\mathcal{O}([X, \rho], [Z, \rho]) = \langle \rho, [X, Z] \rangle, \quad (2.22)$$

where $[X, \rho], [Z, \rho] \in T_\rho \mathcal{O}$ are arbitrary tangent vectors with $X, Z \in \mathfrak{k}$. Now, motivated by the standard ‘shifting trick’ of symplectic reduction, we introduce the product symplectic manifold

$$(\mathcal{P}^{\text{ext}}, \omega^{\text{ext}}) = (\mathcal{P} \times \mathcal{O}, \omega + \omega^\mathcal{O}). \quad (2.23)$$

For an arbitrary function $F \in C^\infty(\mathcal{P}^{\text{ext}})$, at each $u = (y, Y, \rho) \in \mathcal{P}^{\text{ext}}$, we define its gradients

$$\nabla^G F(u) \in \mathfrak{g}, \quad \nabla^\mathfrak{g} F(u) \in \mathfrak{g}, \quad \nabla^\mathcal{O} F(u) \in T_\rho \mathcal{O} \subset \mathfrak{k}, \quad (2.24)$$

by requiring

$$(dF)_u(\delta y \oplus \delta Y \oplus [X, \rho]) = \langle \nabla^G F(u), y^{-1} \delta y \rangle + \langle \nabla^\mathfrak{g} F(u), \delta Y \rangle + \langle \nabla^\mathcal{O} F(u), X \rangle \quad (2.25)$$

for all $\delta y \in T_y G$, $\delta Y \in \mathfrak{g}$ and $X \in \mathfrak{k}$. By combining the definition $\mathbf{X}_F \lrcorner \omega^{\text{ext}} = dF$ with the above formula, for the Hamiltonian vector field $\mathbf{X}_F \in \mathfrak{X}(\mathcal{P}^{\text{ext}})$ we find

$$(\mathbf{X}_F)_u = (y \nabla^\mathfrak{g} F(u))_y \oplus ([Y, \nabla^\mathfrak{g} F(u)] - \nabla^G F(u))_Y \oplus (-\nabla^\mathcal{O} F(u))_\rho. \quad (2.26)$$

Consequently, from the definition $\{F, H\}^{\text{ext}} = \omega^{\text{ext}}(\mathbf{X}_F, \mathbf{X}_H)$ it is immediate that the Poisson bracket of any pair of functions $F, H \in C^\infty(\mathcal{P}^{\text{ext}})$ takes the form

$$\begin{aligned} \{F, H\}^{\text{ext}}(u) &= \langle \nabla^G F(u), \nabla^\mathfrak{g} H(u) \rangle - \langle \nabla^G H(u), \nabla^\mathfrak{g} F(u) \rangle \\ &\quad - \langle [\nabla^\mathfrak{g} F(u), \nabla^\mathfrak{g} H(u)], Y \rangle + \omega_\rho^\mathcal{O}(\nabla^\mathcal{O} F(u), \nabla^\mathcal{O} H(u)). \end{aligned} \quad (2.27)$$

Inspired by the KAK decomposition of G (2.14), let us observe that the map

$$\Phi^{\text{ext}}: (K \times K) \times \mathcal{P}^{\text{ext}} \rightarrow \mathcal{P}^{\text{ext}}, \quad ((k_L, k_R), (y, Y, \rho)) \mapsto (k_L y k_R^{-1}, k_R Y k_R^{-1}, k_L \rho k_L^{-1}) \quad (2.28)$$

is a symplectic left action of $K \times K$ on \mathcal{P}^{ext} , admitting a $K \times K$ -equivariant momentum map

$$J^{\text{ext}}: \mathcal{P}^{\text{ext}} \rightarrow \mathfrak{k} \oplus \mathfrak{k}, \quad (y, Y, \rho) \mapsto ((y Y y^{-1})_+ + \rho) \oplus (-Y_+ - \kappa i \mathbf{C}) \quad (2.29)$$

for all $\kappa \in \mathbb{R}$. As is known (see [8], [9]), the phase space of the hyperbolic BC_n Sutherland model can be derived by reducing the extended phase space \mathcal{P}^{ext} at the *zero value* of the momentum map J^{ext} . In the following we briefly summarize the main steps of the reduction.

First, let \mathfrak{L}_0 denote the set of those points u of the extended phase space (2.23), for which we have $J^{\text{ext}}(u) = 0$. Note that the level set \mathfrak{L}_0 turns out to be an embedded submanifold of \mathcal{P}^{ext} . To analyze its finer structure, we introduce the Lax matrix

$$L: \mathcal{P}^S \rightarrow \mathfrak{g}, \quad (q, p) \mapsto L(q, p) = L_{\mathfrak{p}}(q, p) - \kappa i\mathbf{C}, \quad (2.30)$$

where $L_{\mathfrak{p}}(q, p) \in \mathfrak{p}$ is an Hermitian matrix having the block matrix structure

$$L_{\mathfrak{p}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & -\mathcal{A} \end{bmatrix}. \quad (2.31)$$

More precisely, the entries of the $n \times n$ matrices \mathcal{A} and \mathcal{B} are defined by the formulae

$$\mathcal{A}_{a,b} = \frac{-i\mu}{\sinh(q_a - q_b)}, \quad \mathcal{A}_{c,c} = p_c, \quad \mathcal{B}_{a,b} = \frac{i\mu}{\sinh(q_a + q_b)}, \quad \mathcal{B}_{c,c} = i \frac{\nu + \kappa \cosh(2q_c)}{\sinh(2q_c)}, \quad (2.32)$$

where $a, b, c \in \mathbb{N}_n$, $a \neq b$. We also need the manifold $\mathcal{M}^S = \mathcal{P}^S \times (K \times K)/U(1)_*$, where $U(1)_*$ denotes the diagonal embedding of $U(1)$ into $K \times K$. Now, one can verify that the map

$$\Upsilon^S: \mathcal{M}^S \rightarrow \mathcal{P}^{\text{ext}}, \quad (q, p, (\eta_L, \eta_R)U(1)_*) \mapsto (\eta_L e^Q \eta_R^{-1}, \eta_R L(q, p) \eta_R^{-1}, \eta_L \xi(E) \eta_L^{-1}) \quad (2.33)$$

is an injective immersion with image $\Upsilon^S(\mathcal{M}^S) = \mathfrak{L}_0$. Since the manifolds \mathcal{M}^S and \mathfrak{L}_0 are of the same dimension, this observation leads to the identification $\mathfrak{L}_0 \cong \mathcal{M}^S$.

Second, by examining the (residual) action of $K \times K$ on the model space \mathcal{M}^S of the level set \mathfrak{L}_0 , it is immediate that the base manifold of the trivial principal $(K \times K)/U(1)_*$ -bundle

$$\pi^S: \mathcal{M}^S \twoheadrightarrow \mathcal{P}^S, \quad (q, p, (\eta_L, \eta_R)U(1)_*) \mapsto (q, p) \quad (2.34)$$

provides a convenient model for the reduced symplectic manifold. That is, we end up with the natural identifications

$$\mathcal{P}^{\text{ext}} //_0 (K \times K) \cong \mathcal{M}^S / (K \times K) \cong \mathcal{P}^S. \quad (2.35)$$

Making use of the defining relationship $(\pi^S)^* \omega^{\text{red}} = (\Upsilon^S)^* \omega^{\text{ext}}$, for the reduced symplectic form we find immediately that $\omega^{\text{red}} = 2\omega^S$ with the canonical symplectic form ω^S (1.3). Consequently, for the reduced Poisson bracket we obtain

$$\{\cdot, \cdot\}^S = 2\{\cdot, \cdot\}^{\text{red}}. \quad (2.36)$$

Finally, let us consider the $K \times K$ -invariant quadratic Hamiltonian

$$F_2(y, Y, \rho) = \frac{1}{4} \langle Y, Y \rangle = \frac{1}{4} \text{tr}(Y^2) \quad ((y, Y, \rho) \in \mathcal{P}^{\text{ext}}). \quad (2.37)$$

It is clear that F_2 generates the ‘free’ geodesic motion on the group manifold G . Due to its invariance, it survives the reduction and the corresponding reduced Hamiltonian coincides with the Hamiltonian of the Sutherland model (1.4) with coupling constants

$$g^2 = \mu^2, \quad g_1^2 = \frac{1}{2} \nu \kappa, \quad g_2^2 = \frac{1}{2} (\nu - \kappa)^2. \quad (2.38)$$

Just now can we really appreciate the inclusion of the innocent looking κ -dependent central element $\kappa i\mathbf{C}$ into the momentum map J^{ext} (2.29). Indeed, by specializing the parameters (μ, ν, κ) appropriately, from the proposed reduction picture we can recover the most general hyperbolic BC_n Sutherland model with *three* independent coupling constants.

3 Construction of the r -matrix

Since the eigenvalues of L (2.30) are in involution (see [9]), we know from general principles that the Lax matrix obeys an r -matrix Poisson bracket. As is known (see e.g. [11]), there is a general, purely algebraic approach to find an explicit formula for the r -matrix. However, we rather follow the symplectic reduction approach put forward by Avan, Babelon and Talon in [10]. It is worth mentioning that this geometric approach was later generalized and systematically exploited in [12], leading to a uniform treatment of the classical r -matrix structure for various integrable systems.

3.1 Local extensions of the Lax matrix

Take an arbitrary point (q, p) of \mathcal{P}^S and keep it fixed. Notice that the point

$$z_0 = (q, p, (\mathbf{1}_N, \mathbf{1}_N)U(1)_*) \in \mathcal{M}^S \quad (3.1)$$

projects onto (q, p) , i.e. $\pi^S(z_0) = (q, p)$. Now, pick an arbitrary function $f \in C^\infty(\mathcal{P}^S)$. We say that a smooth function

$$\tilde{f}: U \rightarrow \mathbb{R} \quad (3.2)$$

defined on some open neighborhood $U \subset \mathcal{P}^{\text{ext}}$ of point

$$u_0 = \Upsilon^S(z_0) = (e^Q, L(q, p), \xi(E)) \in \mathcal{P}^{\text{ext}} \quad (3.3)$$

is a *local extension* of f around u_0 , if

$$\tilde{f} \circ \Upsilon^S|_{(\Upsilon^S)^{-1}(U)} = f \circ \pi^S|_{(\Upsilon^S)^{-1}(U)}. \quad (3.4)$$

As is known, this special class of local extensions can be used effectively to compute reduced Poisson brackets by evaluating certain ‘unreduced’ Poisson brackets. More precisely, if $\tilde{f}, \tilde{g} \in C^\infty(U)$ are arbitrary local extensions of functions $f, g \in C^\infty(\mathcal{P}^S)$ around u_0 in the sense of (3.4), then

$$\{f, g\}^{\text{red}}(q, p) = \{\tilde{f}, \tilde{g}\}^{\text{ext}}(u_0). \quad (3.5)$$

In particular, in the following we shall make use of the above formula¹ to find an explicit expression for the r -matrix of the Sutherland model. The auxiliary functions defined below play an important role in constructing local extensions of the Lax matrix L (2.30).

We start with the study of the smooth principal M -bundle π (2.15) induced by the KAK decomposition of G . Since $\pi(q, \mathbf{1}_N, \mathbf{1}_N) = e^Q$, there is a smooth local section

$$\check{G} \ni y \mapsto (\sigma_{\mathfrak{c}}(y), \sigma_L(y), \sigma_R(y)) \in \mathfrak{c} \times K \times K \quad (3.6)$$

of π , defined on some small open neighborhood $\check{G} \subset G_{\text{reg}}$ of e^Q , such that

$$(\sigma_{\mathfrak{c}}(e^Q), \sigma_L(e^Q), \sigma_R(e^Q)) = (q, \mathbf{1}_N, \mathbf{1}_N). \quad (3.7)$$

Besides the above normalization, we may impose certain conditions on the derivative of section (3.6) at e^Q , too. Notice that the tangent space of $\mathfrak{c} \times K \times K$ at $(q, \mathbf{1}_N, \mathbf{1}_N)$ can be identified as

$$T_{(q, \mathbf{1}_N, \mathbf{1}_N)}(\mathfrak{c} \times K \times K) \cong T_q \mathfrak{c} \oplus T_{\mathbf{1}_N} K \oplus T_{\mathbf{1}_N} K \cong \mathbb{R}^n \oplus \mathfrak{k} \oplus \mathfrak{k}, \quad (3.8)$$

¹ Consistently with the Dirac bracket, generally there are also some correction terms on the right hand side of (3.5). However, since we reduce at the *zero value* of the (equivariant) momentum map, and since by (3.4) our local extensions are (locally) $K \times K$ -invariant on the level set \mathfrak{L}_0 , these corrections cancel. For details see e.g. Chapter 14 in [13].

in which the vertical subspace of π takes the form

$$\ker((d\pi)_{(q, \mathbf{1}_N, \mathbf{1}_N)}) = \{0 \oplus X \oplus X \in \mathbb{R}^n \oplus \mathfrak{k} \oplus \mathfrak{k} \mid X \in \mathfrak{m}\} \cong \mathfrak{m}. \quad (3.9)$$

Since $\mathbb{R}^n \oplus \mathfrak{m}^\perp \oplus \mathfrak{k}$ is clearly a complementary subspace of the vertical subspace, we may assume that at point e^Q the derivative of (3.6) maps into the complementary ‘horizontal’ subspace, i.e.

$$\text{ran}((d(\sigma_{\mathfrak{c}}, \sigma_L, \sigma_R))_{e^Q}) = \mathbb{R}^n \oplus \mathfrak{m}^\perp \oplus \mathfrak{k}. \quad (3.10)$$

That is, we may assume that

$$\text{ran}((d\sigma_L)_{e^Q}) = \mathfrak{m}^\perp. \quad (3.11)$$

In the following we will need an explicit formula for the derivative of (3.6) at point e^Q .

Lemma 1. *Under assumption (3.11), for each tangent vector $\delta Y \in \mathfrak{g} \cong T_{\mathbf{1}_N}G$ we have*

$$(d\sigma_L)_{e^Q}(e^Q \delta Y) = -\sinh(\widetilde{\text{ad}}_Q)^{-1}(\delta Y)_{\mathfrak{a}^\perp}, \quad (3.12)$$

$$(d\sigma_R)_{e^Q}(e^Q \delta Y) = -(\delta Y)_+ - \coth(\widetilde{\text{ad}}_Q)(\delta Y)_{\mathfrak{a}^\perp}. \quad (3.13)$$

Proof. For simplicity, let us introduce the shorthand notations

$$\delta_{\mathfrak{c}} = (d\sigma_{\mathfrak{c}})_{e^Q}(e^Q \delta Y) \in \mathbb{R}^n, \quad \delta_L = (d\sigma_L)_{e^Q}(e^Q \delta Y) \in \mathfrak{m}^\perp, \quad \delta_R = (d\sigma_R)_{e^Q}(e^Q \delta Y) \in \mathfrak{k}, \quad (3.14)$$

and define $D_{\mathfrak{c}} = \text{diag}(\delta_{\mathfrak{c}}, -\delta_{\mathfrak{c}}) \in \mathfrak{a}$. Since \check{G} is open, for small values of $|t|$ we have $e^Q e^{t\delta Y} \in \check{G}$, whence

$$e^Q e^{t\delta Y} = \sigma_L(e^Q e^{t\delta Y}) e^{\text{diag}(\sigma_{\mathfrak{c}}(e^Q e^{t\delta Y}), -\sigma_{\mathfrak{c}}(e^Q e^{t\delta Y}))} \sigma_R(e^Q e^{t\delta Y})^{-1}. \quad (3.15)$$

By taking the derivative of the above equation at $t = 0$, we obtain

$$\delta Y = \cosh(\widetilde{\text{ad}}_Q)\delta_L - \sinh(\widetilde{\text{ad}}_Q)\delta_L + D_{\mathfrak{c}} - \delta_R. \quad (3.16)$$

It follows that $(\delta Y)_{\mathfrak{a}} = D_{\mathfrak{c}}$ and $(\delta Y)_{\mathfrak{m}} = -(\delta_R)_{\mathfrak{m}}$, meanwhile for the off-diagonal components we get

$$(\delta Y)_{\mathfrak{a}^\perp} = -\sinh(\widetilde{\text{ad}}_Q)\delta_L, \quad (\delta Y)_{\mathfrak{m}^\perp} = \cosh(\widetilde{\text{ad}}_Q)\delta_L - (\delta_R)_{\mathfrak{m}^\perp}. \quad (3.17)$$

By solving this linear system for $D_{\mathfrak{c}}$, δ_L and δ_R , the lemma follows. \square

To proceed further, let us note that

$$\check{S} = \{V \in S \mid V_1 \neq 0, \dots, V_n \neq 0\} \quad (3.18)$$

is an open and dense submanifold of S (2.16), which contains E (2.18). The map

$$\tau: \check{S} \rightarrow M, \quad V \mapsto \text{diag}\left(\frac{V_1}{|V_1|}, \dots, \frac{V_n}{|V_n|}, \frac{V_1}{|V_1|}, \dots, \frac{V_n}{|V_n|}\right) \quad (3.19)$$

defined on \check{S} is smooth, satisfying $\tau(E) = \mathbf{1}_N$. Utilizing the natural identification

$$T_E S \cong \{\delta V \in \mathbb{C}^N \mid \mathcal{C}\delta V + \delta V = 0, (\delta V)^* E + E^* \delta V = 0\}, \quad (3.20)$$

for the action of the derivative of τ on each tangent vector $\delta V \in T_E S$ we find

$$(d\tau)_E(\delta V) = i \text{diag}(\text{Im}((\delta V)_1), \dots, \text{Im}((\delta V)_n), \text{Im}((\delta V)_1), \dots, \text{Im}((\delta V)_n)) \in \mathfrak{m}. \quad (3.21)$$

Turning to the principal $U(1)$ -bundle ξ (2.20), notice that E projects onto $\xi(E)$. Therefore, we can find a smooth local section

$$\check{\mathcal{O}} \ni \rho \mapsto \mathcal{V}(\rho) \in S \quad (3.22)$$

of ξ , defined on some open neighborhood $\check{\mathcal{O}} \subset \mathcal{O}$ of $\xi(E)$, such that $\mathcal{V}(\xi(E)) = E$. Moreover, by ‘shrinking’ $\check{\mathcal{O}}$ if necessary, we may assume that

$$\mathcal{V}(\rho) \in \check{S} \quad (\forall \rho \in \check{\mathcal{O}}). \quad (3.23)$$

In order to fix the range of the derivative of section \mathcal{V} at point $\xi(E)$, notice that the map

$$T_E S \times T_E S \ni (\delta V, \delta W) \mapsto \operatorname{Re}((\delta V)^* \delta W) \in \mathbb{R} \quad (3.24)$$

is an inner product on the tangent space $T_E S$ (3.20). Therefore, we may assume that the derivative operator $(d\mathcal{V})_{\xi(E)}$ maps $T_{\xi(E)} \mathcal{O}$ into the orthogonal complement of the vertical subspace

$$\ker((d\xi)_E) = \mathbb{R}iE = \{xiE \in T_E S \mid x \in \mathbb{R}\}. \quad (3.25)$$

This requirement amounts to the constraint

$$(\delta V)^* E = E^* \delta V \quad (\forall \delta V \in \operatorname{ran}((d\mathcal{V})_{\xi(E)})). \quad (3.26)$$

Remembering the local section (3.6), let us consider the smooth map

$$\gamma: \check{G} \times \check{\mathcal{O}} \rightarrow \mathcal{O}, \quad (y, \rho) \mapsto \sigma_L(y)^{-1} \rho \sigma_L(y). \quad (3.27)$$

Notice that $\gamma(e^Q, \xi(E)) = \xi(E) \in \check{\mathcal{O}}$. Therefore, there are some open neighborhoods $\hat{G} \subset \check{G}$ of e^Q , and $\hat{\mathcal{O}} \subset \check{\mathcal{O}}$ of $\xi(E)$, such that

$$\gamma(y, \rho) \in \check{\mathcal{O}} \quad (\forall (y, \rho) \in \hat{G} \times \hat{\mathcal{O}}). \quad (3.28)$$

Having equipped with τ , \mathcal{V} and γ , now we define their composition

$$m: \hat{G} \times \hat{\mathcal{O}} \rightarrow M, \quad (y, \rho) \mapsto m(y, \rho) = \tau(\mathcal{V}(\gamma(y, \rho))), \quad (3.29)$$

which is a smooth map satisfying $m(e^Q, \xi(E)) = \mathbf{1}_N$. To compute the derivatives of the matrix entries of the diagonal matrix

$$m(y, \rho) = \operatorname{diag}(m_1(y, \rho), \dots, m_n(y, \rho), m_1(y, \rho), \dots, m_n(y, \rho)) \quad (3.30)$$

at point $(e^Q, \xi(E))$, for each $c \in \mathbb{N}_n$ we introduce the column vector $F_c \in \mathbb{C}^N$ with components

$$(F_c)_a = -(F_c)_{n+a} = \delta_{c,a} \quad (a \in \mathbb{N}_n), \quad (3.31)$$

together with the Lie algebra element

$$\Xi_c = i \left(F_c E^* + E F_c^* - \frac{2}{n} E E^* \right) \in \mathfrak{k}. \quad (3.32)$$

Lemma 2. *Take arbitrary vectors $\delta Y \in \mathfrak{g}$ and $Z \in \mathfrak{k}$; then for each $c \in \mathbb{N}_n$ we have*

$$(dm_c)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) = -\frac{i}{4} \left\langle \Xi_c, Z + \sinh(\widetilde{\operatorname{ad}}_Q)^{-1}(\delta Y)_{\mathfrak{a}^\perp} \right\rangle. \quad (3.33)$$

Proof. From (3.29) it is clear that

$$(\mathrm{d}m)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) = (\mathrm{d}\tau)_E(\mathrm{d}\mathcal{V})_{\xi(E)}(\mathrm{d}\gamma)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]). \quad (3.34)$$

Upon introducing the shorthand notation

$$X = Z + \sinh(\widetilde{\mathrm{ad}}_Q)^{-1}(\delta Y)_{\mathfrak{a}^\perp} \in \mathfrak{k}, \quad (3.35)$$

from (3.27) and Lemma 1 it is immediate that

$$(\mathrm{d}\gamma)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) = [X, \xi(E)] \in T_{\xi(E)}\mathcal{O}. \quad (3.36)$$

Next, let us consider the tangent vector

$$\delta V = (\mathrm{d}\mathcal{V})_{\xi(E)}(\mathrm{d}\gamma)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) \in T_E S. \quad (3.37)$$

Since \mathcal{V} (3.22) is a local section, we have $\xi \circ \mathcal{V} = \mathrm{Id}_{\tilde{\mathcal{O}}}$, which entails

$$(\mathrm{d}\xi)_E \delta V = (\mathrm{d}\xi)_E(\mathrm{d}\mathcal{V})_{\xi(E)}[X, \xi(E)] = [X, \xi(E)]. \quad (3.38)$$

On the other hand, notice that the vector XE belongs to the tangent space $T_E S$ (3.20). Moreover, we find easily that

$$(\mathrm{d}\xi)_E(XE) = [X, \xi(E)]. \quad (3.39)$$

From the last two equations we conclude that $\delta V - XE$ belongs to the vertical subspace (3.25), whence $\delta V = XE + xiE$ with some $x \in \mathbb{R}$. However, the value of x is uniquely determined by (3.26), from where we infer that

$$\delta V = XE - \frac{E^* X E}{N} E. \quad (3.40)$$

Now, by combining (3.34) and (3.37), we see that

$$(\mathrm{d}m)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) = (\mathrm{d}\tau)_E \delta V. \quad (3.41)$$

Therefore, recalling (3.21) and (3.30), for each $c \in \mathbb{N}_n$ we can write

$$(\mathrm{d}m_c)_{(e^Q, \xi(E))}(e^Q \delta Y \oplus [Z, \xi(E)]) = \mathrm{i} \operatorname{Im}((\delta V)_c) = \mathrm{i} \operatorname{Im}((XE)_c) - \frac{\operatorname{tr}(X E E^*)}{N}. \quad (3.42)$$

Utilizing (3.31), notice that

$$\mathrm{i} \operatorname{Im}((XE)_c) = \frac{1}{2} \mathrm{i} \operatorname{Im}(F_c^* X E) = \frac{1}{4} (F_c^* X E + E^* X F_c) = \frac{1}{4\mathrm{i}} \operatorname{tr}(\mathrm{i}(F_c E^* + E F_c^*) X). \quad (3.43)$$

Plugging this formula into (3.42), the lemma follows. \square

Now we are in a position to construct an appropriate local extension of the Lax operator L (2.30). For this, let us consider the open subset

$$\hat{\mathcal{P}}^{\mathrm{ext}} = \hat{G} \times \mathfrak{g} \times \hat{\mathcal{O}} \subset \mathcal{P}^{\mathrm{ext}}, \quad (3.44)$$

which clearly contains the reference point u_0 (3.3). Recalling (3.6) and (3.29), on $\hat{\mathcal{P}}^{\mathrm{ext}}$ we define the smooth function

$$\varphi: \hat{\mathcal{P}}^{\mathrm{ext}} \rightarrow K, \quad (y, Y, \rho) \mapsto \varphi(y, Y, \rho) = \sigma_R(y) m(y, \rho). \quad (3.45)$$

Notice that $\varphi(u_0) = \mathbf{1}_N$. Finally, let us consider the locally defined smooth function

$$\tilde{L}: \hat{\mathcal{P}}^{\mathrm{ext}} \rightarrow \mathfrak{g}, \quad (y, Y, \rho) \mapsto \tilde{L}(y, Y, \rho) = \varphi(y, Y, \rho)^{-1} Y \varphi(y, Y, \rho). \quad (3.46)$$

Lemma 3. *The \mathfrak{g} -valued function \tilde{L} (3.46), defined on a small open neighborhood of the reference point u_0 (3.3), is a local extension of L (2.30). More precisely, we have*

$$\tilde{L} \circ \Upsilon^S|_{(\Upsilon^S)^{-1}(\hat{\mathcal{P}}^{\text{ext}})} = L \circ \pi^S|_{(\Upsilon^S)^{-1}(\hat{\mathcal{P}}^{\text{ext}})}. \quad (3.47)$$

Proof. Take an arbitrary point $\tilde{z} = (\tilde{q}, \tilde{p}, (\tilde{\eta}_L, \tilde{\eta}_R)U(1)_*) \in (\Upsilon^S)^{-1}(\hat{\mathcal{P}}^{\text{ext}})$, and let

$$(\tilde{y}, \tilde{Y}, \tilde{\rho}) = \Upsilon^S(\tilde{z}) = (\tilde{\eta}_L e^{\tilde{Q}} \tilde{\eta}_R^{-1}, \tilde{\eta}_R L(\tilde{q}, \tilde{p}) \tilde{\eta}_R^{-1}, \tilde{\eta}_L \xi(E) \tilde{\eta}_L^{-1}) \in \hat{\mathcal{P}}^{\text{ext}}. \quad (3.48)$$

By applying the local section σ (3.6) on the Lie group element $\tilde{y} \in \hat{G} \subset \check{G}$, we see that

$$\tilde{y} = \sigma_L(\tilde{y}) e^{\text{diag}(\sigma_{\mathfrak{c}}(\tilde{y}), -\sigma_{\mathfrak{c}}(\tilde{y}))} \sigma_R(\tilde{y})^{-1}. \quad (3.49)$$

Recalling π (2.15), it is immediate that $\tilde{q} = \sigma_{\mathfrak{c}}(\tilde{y})$ and $(\tilde{\eta}_L, \tilde{\eta}_R)M_* = (\sigma_L(\tilde{y}), \sigma_R(\tilde{y}))M_*$. Therefore, we have

$$\tilde{\eta}_L = \sigma_L(\tilde{y})\mathcal{D} \quad \text{and} \quad \tilde{\eta}_R = \sigma_R(\tilde{y})\mathcal{D} \quad (3.50)$$

with some diagonal matrix $\mathcal{D} \in M$.

Next, we inspect the Lie group element $m(\tilde{y}, \tilde{\rho}) = \tau(\mathcal{V}(\gamma(\tilde{y}, \tilde{\rho}))) \in M$. Remembering (3.27), notice that

$$\gamma(\tilde{y}, \tilde{\rho}) = \sigma_L(\tilde{y})^{-1} \tilde{\rho} \sigma_L(\tilde{y}) = \sigma_L(\tilde{y})^{-1} \tilde{\eta}_L \xi(E) \tilde{\eta}_L^{-1} \sigma_L(\tilde{y}) = \mathcal{D} \xi(E) \mathcal{D}^{-1} = \xi(\mathcal{D}E). \quad (3.51)$$

Since \mathcal{V} (3.22) is a local section of ξ (2.20), we find that

$$\xi(\mathcal{V}(\gamma(\tilde{y}, \tilde{\rho}))) = (\xi \circ \mathcal{V})(\xi(\mathcal{D}E)) = \xi(\mathcal{D}E). \quad (3.52)$$

Therefore, there is some $\psi \in \mathbb{R}$, such that $\mathcal{V}(\gamma(\tilde{y}, \tilde{\rho})) = e^{i\psi} \mathcal{D}E$. Recalling (3.19), it follows that

$$m(\tilde{y}, \tilde{\rho}) = \tau(e^{i\psi} \mathcal{D}E) = e^{i\psi} \mathcal{D}. \quad (3.53)$$

Due to relationships (3.50) and (3.53) we observe that

$$\varphi(\tilde{y}, \tilde{Y}, \tilde{\rho}) = \sigma_R(\tilde{y}) m(\tilde{y}, \tilde{\rho}) = e^{i\psi} \tilde{\eta}_R. \quad (3.54)$$

Therefore, recalling (3.46) and (3.48), we find immediately that

$$\tilde{L}(\Upsilon^S(\tilde{z})) = \varphi(\tilde{y}, \tilde{Y}, \tilde{\rho})^{-1} \tilde{Y} \varphi(\tilde{y}, \tilde{Y}, \tilde{\rho}) = L(\tilde{q}, \tilde{p}) = L(\pi^S(\tilde{z})). \quad (3.55)$$

Since \tilde{z} is an arbitrary element of $(\Upsilon^S)^{-1}(\hat{\mathcal{P}}^{\text{ext}})$, the lemma follows. \square

3.2 Computing the r -matrix

Let us choose some dual bases $\{T_A\}$, $\{T^A\}$ in \mathfrak{g} , i.e. $\langle T^A, T_B \rangle = \delta_B^A$, and consider the function

$$\mathcal{P}^{\text{ext}} \ni (y, Y, \rho) \mapsto \mathbf{Y}(y, Y, \rho) = Y \in \mathfrak{g}, \quad (3.56)$$

together with its components

$$\mathcal{P}^{\text{ext}} \ni (y, Y, \rho) \mapsto \mathbf{Y}^A(y, Y, \rho) = \langle T^A, Y \rangle \in \mathbb{R}, \quad (3.57)$$

defined on the extended phase space. Notice that $\mathbf{Y} = \sum_A \mathbf{Y}^A T_A$. Since \mathbf{Y}^A depends only on variable Y , its only nontrivial gradient (2.24) is

$$\nabla^{\mathfrak{g}} \mathbf{Y}^A \equiv T^A, \quad (3.58)$$

whence from (2.27) we obtain that $\{\mathbf{Y}^A, \mathbf{Y}^B\}^{\text{ext}} = -\langle [T^A, T^B], \mathbf{Y} \rangle$. As usual in the theory of integrable systems, these relationships can be succinctly rewritten in the standard St Petersburg tensorial² notation. Indeed, upon introducing the quadratic Casimir

$$\Omega_{12} = \Omega_{21} = \sum_A T_A \otimes T^A \in \mathfrak{g} \otimes \mathfrak{g}, \quad (3.59)$$

we find easily that

$$\{\mathbf{Y}_1, \mathbf{Y}_2\}^{\text{ext}} = \sum_{A,B} \{\mathbf{Y}^A, \mathbf{Y}^B\}^{\text{ext}} T_A \otimes T_B = [-\Omega_{12}/2, \mathbf{Y}_1] - [-\Omega_{21}/2, \mathbf{Y}_2]. \quad (3.60)$$

Now, from (3.46) and (3.56) we see that \tilde{L} can be obtained from \mathbf{Y} by the gauge transformation

$$\tilde{L} = \varphi^{-1} \mathbf{Y} \varphi. \quad (3.61)$$

It readily follows³ that $\{\tilde{L}_1, \tilde{L}_2\}^{\text{ext}} = [\tilde{r}_{12}, \tilde{L}_1] - [\tilde{r}_{21}, \tilde{L}_2]$ with the transformed r -matrix

$$\tilde{r}_{12} = \varphi_1^{-1} \varphi_2^{-1} \left(-\frac{1}{2} \Omega_{12} - \{\varphi_1, \mathbf{Y}_2\}^{\text{ext}} \varphi_1^{-1} + \frac{1}{2} [\{\varphi_1, \varphi_2\}^{\text{ext}} \varphi_1^{-1} \varphi_2^{-1}, \mathbf{Y}_2] \right) \varphi_1 \varphi_2. \quad (3.62)$$

Recalling (2.36) and (3.5), it is thus immediate that for the Lax matrix L (2.30) we have

$$\{L_1, L_2\}^S(q, p) = [r_{12}(q, p), L_1(q, p)] - [r_{21}(q, p), L_2(q, p)] \quad (3.63)$$

with the r -matrix

$$r_{12}(q, p) = 2\tilde{r}_{12}(u_0) = -\Omega_{12} - 2\{\varphi_1, \mathbf{Y}_2\}^{\text{ext}}(u_0) + [\{\varphi_1, \varphi_2\}^{\text{ext}}(u_0), L_2(q, p)]. \quad (3.64)$$

In the following we use extensively the special basis of \mathfrak{g} introduced in the appendix. As a first step, we define the Lie algebra elements

$$Z_{e_a \pm e_b} = \frac{D_a^+ + D_b^+}{\sqrt{2}} \in \mathfrak{m} \quad \text{and} \quad Z_{2e_c} = D_c^+ \in \mathfrak{m}, \quad (3.65)$$

where $a, b, c \in \mathbb{N}_n$ and $a < b$. Observe that these matrices are labeled by the C_n -type positive roots (A.2). Now, we can formulate the main result of the paper.

Theorem 4. *The Lax matrix (2.30) of the classical hyperbolic BC_n Sutherland model verifies the r -matrix Poisson bracket (3.63) with the q -dependent r -matrix*

$$\begin{aligned} r_{12}(q) = & 2 \sum_{\alpha, \epsilon} \coth(\alpha(q)) X_{\alpha}^{+, \epsilon} \otimes X_{\alpha}^{-, \epsilon} - 2 \sum_{\alpha} \frac{1}{\sinh(\alpha(q))} Z_{\alpha} \otimes X_{\alpha}^{-, i} \\ & - \sum_c (D_c^+ \otimes D_c^+ + D_c^- \otimes D_c^-) - \sum_{\alpha, \epsilon} (X_{\alpha}^{+, \epsilon} \otimes X_{\alpha}^{+, \epsilon} + X_{\alpha}^{-, \epsilon} \otimes X_{\alpha}^{-, \epsilon}). \end{aligned} \quad (3.66)$$

² We write $L_1 = L \otimes \mathbf{1}$, $L_2 = \mathbf{1} \otimes L$, together with $r_{12} = \sum r^{A,B} T_A \otimes T_B$, $r_{21} = \sum r^{A,B} T_B \otimes T_A$, etc.

³ For details on gauge transformations see e.g. [14], or Chapter 2 in [13].

Proof. As dictated by (3.64), we inspect the formula of $r_{12}(q, p)$ term-by-term. Using the basis given in the appendix, it is clear that the quadratic Casimir (3.59) takes the form

$$\Omega_{12} = \sum_c (D_c^- \otimes D_c^- - D_c^+ \otimes D_c^+) + \sum_{\alpha, \epsilon} (X_\alpha^{-, \epsilon} \otimes X_\alpha^{-, \epsilon} - X_\alpha^{+, \epsilon} \otimes X_\alpha^{+, \epsilon}). \quad (3.67)$$

Recalling (2.27) and (3.45), it is also clear that

$$\{\varphi_1, \mathbf{Y}_2\}^{\text{ext}}(u_0) = \sum_A \left((d\sigma_R)_{eQ} (e^Q T^A) + (dm)_{(eQ, \xi(E))} (e^Q T^A \oplus 0) \right) \otimes T_A. \quad (3.68)$$

Now, from Lemma 1 it is immediate that

$$\sum_A (d\sigma_R)_{eQ} (e^Q T^A) \otimes T_A = \sum_c D_c^+ \otimes D_c^+ + \sum_{\alpha, \epsilon} X_\alpha^{+, \epsilon} \otimes (X_\alpha^{+, \epsilon} - \coth(\alpha(q)) X_\alpha^{-, \epsilon}), \quad (3.69)$$

whereas Lemma 2 leads to the formula

$$\sum_A (dm)_{(eQ, \xi(E))} (e^Q T^A \oplus 0) \otimes T_A = -\frac{\sqrt{2}}{4} \sum_A \sum_c \left\langle \Xi_c, \sinh(\widetilde{\text{ad}}_Q)^{-1}(T^A)_{\mathfrak{a}^\perp} \right\rangle D_c^+ \otimes T_A. \quad (3.70)$$

Remembering definition (3.32), notice that the c -dependent part of Ξ_c can be rewritten as

$$\mathfrak{i}(F_c E^* + E F_c^*) = 2\sqrt{2}(D_c^+ + X_{2e_c}^{+, \mathfrak{i}}) + 2 \sum_{d=1}^{c-1} (X_{e_d - e_c}^{+, \mathfrak{i}} + X_{e_d + e_c}^{+, \mathfrak{i}}) + 2 \sum_{d=c+1}^n (X_{e_c - e_d}^{+, \mathfrak{i}} + X_{e_c + e_d}^{+, \mathfrak{i}}). \quad (3.71)$$

Therefore, upon introducing the Lie algebra element

$$\Psi(q) = \frac{1}{N} \sum_A \left\langle \mathfrak{i} E E^*, \sinh(\widetilde{\text{ad}}_Q)^{-1}(T^A)_{\mathfrak{a}^\perp} \right\rangle T_A \in \mathfrak{g}, \quad (3.72)$$

we find easily that

$$\sum_A (dm)_{(eQ, \xi(E))} (e^Q T^A \oplus 0) \otimes T_A = \mathfrak{i} \mathbf{1}_N \otimes \Psi(q) + \sum_\alpha \frac{1}{\sinh(\alpha(q))} Z_\alpha \otimes X_\alpha^{-, \mathfrak{i}}. \quad (3.73)$$

By plugging formulae (3.69) and (3.73) into (3.68), the control over $\{\varphi_1, \mathbf{Y}_2\}^{\text{ext}}(u_0)$ is complete.

To proceed further, let us introduce the locally defined smooth functions

$$\tilde{\sigma}_R(y, Y, \rho) = \sigma_R(y) \quad \text{and} \quad \tilde{m}(y, Y, \rho) = m(y, \rho) \quad ((y, Y, \rho) \in \hat{\mathcal{P}}^{\text{ext}}). \quad (3.74)$$

Due to (3.45) it is clear that $\varphi = \tilde{\sigma}_R \tilde{m}$. Now, from (2.27) we see that on $\hat{\mathcal{P}}^{\text{ext}}$ we have

$$\{(\tilde{\sigma}_R)_1, (\tilde{\sigma}_R)_2\}^{\text{ext}} \equiv 0, \quad \{(\tilde{\sigma}_R)_1, (\tilde{m})_2\}^{\text{ext}} \equiv 0, \quad \{(\tilde{m})_1, (\tilde{\sigma}_R)_2\}^{\text{ext}} \equiv 0, \quad (3.75)$$

therefore $\{\varphi_1, \varphi_2\}^{\text{ext}}(u_0) = \{(\tilde{m})_1, (\tilde{m})_2\}^{\text{ext}}(u_0)$ readily follows. Keeping our focus on this relationship, from (2.25) and Lemma 2 it is immediate that

$$\nabla^\mathcal{O} \text{Re}(\tilde{m}_c)(u_0) = 0 \quad \text{and} \quad \nabla^\mathcal{O} \text{Im}(\tilde{m}_c)(u_0) = -\Xi_c/4. \quad (3.76)$$

Note that the Lie algebra element Ξ_c (3.32) can be represented as an appropriate commutator. Indeed, upon introducing the matrices

$$A_c = \frac{1}{n} \sum_{d=1}^n (e_{c,d} - e_{d,c}) \in \mathfrak{u}(n) \quad \text{and} \quad V_c = \text{diag}(A_c, A_c) \in \mathfrak{k}, \quad (3.77)$$

we find immediately that $\Xi_c = \mu^{-1}[V_c, \xi(E)]$. Therefore, from (2.27) and (2.22) we obtain that

$$\{\tilde{m}_c, \tilde{m}_d\}^{\text{ext}}(u_0) = -\frac{1}{16\mu^2} \langle \xi(E), [V_c, V_d] \rangle = 0, \quad (3.78)$$

for all $c, d \in \mathbb{N}_n$. Thus, we end up with the simple relationship $\{\varphi_1, \varphi_2\}^{\text{ext}}(u_0) = 0$.

We conclude the proof with the observation that the term $i\mathbf{1}_N \otimes \Psi(q)$ appearing in (3.73) can be neglected, since it commutes with $L_1(q, p)$. Therefore, by simply plugging the above derived formulae into (3.64), the theorem follows. \square

Switching to the standard basis $\{e_{k,l}\}$ of the matrix Lie algebra $\mathfrak{gl}(N, \mathbb{C})$, the r -matrix (3.66) can be rewritten as

$$\begin{aligned} r_{12}(q) = & \sum_{\substack{a,b=1 \\ (a \neq b)}}^n \coth(q_a - q_b) (e_{a,b} + e_{n+a,n+b}) \otimes (e_{b,a} - e_{n+b,n+a}) \\ & + \sum_{a,b=1}^n \coth(q_a + q_b) (e_{a,n+b} + e_{n+a,b}) \otimes (e_{n+b,a} - e_{b,n+a}) \\ & + \frac{1}{2} \sum_{\substack{a,b=1 \\ (a \neq b)}}^n \frac{1}{\sinh(q_a - q_b)} (e_{a,a} + e_{n+a,n+a} + e_{b,b} + e_{n+b,n+b}) \otimes (e_{a,b} - e_{n+a,n+b}) \\ & - \frac{1}{2} \sum_{a,b=1}^n \frac{1}{\sinh(q_a + q_b)} (e_{a,a} + e_{n+a,n+a} + e_{b,b} + e_{n+b,n+b}) \otimes (e_{a,n+b} - e_{n+a,b}) \\ & + \sum_{a,b=1}^n (e_{a,b} \otimes e_{n+b,n+a} + e_{n+a,n+b} \otimes e_{b,a} + e_{a,n+b} \otimes e_{b,n+a} + e_{n+a,b} \otimes e_{n+b,a}). \end{aligned} \quad (3.79)$$

To conclude this subsection, notice that the above r -matrix can be seen as a generalization of the C_n -type r -matrix constructed by Avan, Babelon and Talon. Indeed, up to a constant conjugation, the q -dependent part of (3.79) can be identified with the r -matrix of the C_n Sutherland model presented in [10]. Nevertheless, as one can easily verify by inspecting the r -matrix Poisson bracket (3.63), in the special case $\kappa = 0$ the q -independent part of (3.79) can be safely neglected. In other words, with the specialization $\kappa = 0$ we can also recover the C_n -type r -matrix of paper [10].

3.3 Lax representation of the dynamics

Having constructed an r -matrix for the BC_n Sutherland model, we can automatically provide a Lax representation for the dynamics as well. For this, we need the operator version of $r_{12}(q)$ (3.66), which is defined via the natural identifications

$$\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \text{End}(\mathfrak{g}). \quad (3.80)$$

More precisely, the linear operator $R(q) \in \text{End}(\mathfrak{g})$ corresponding to the element $r_{12}(q) \in \mathfrak{g} \otimes \mathfrak{g}$ can be recovered from the formula

$$R(q)Y = \text{tr}_2(r_{12}(q)Y_2) \quad (Y \in \mathfrak{g}), \quad (3.81)$$

where the linear operator tr_2 defined by $\text{tr}_2(X \otimes Y) = \text{tr}(Y)X$ is the usual partial trace on the second factor. From (3.66) and (3.81) it is immediate that

$$R(q)Y = 2 \sum_{\alpha, \epsilon} \coth(\alpha(q)) \langle X_{\alpha}^{-, \epsilon}, Y \rangle X_{\alpha}^{+, \epsilon} - 2 \sum_{\alpha} \frac{1}{\sinh(\alpha(q))} \langle X_{\alpha}^{-, i}, Y \rangle Z_{\alpha} - Y^*. \quad (3.82)$$

Now, let us introduce the matrix-valued function $B = \frac{1}{2}(L + RL)$ defined on the phase space \mathcal{P}^S . Since the Lax matrix L (2.30) can be written as

$$\begin{aligned} L(q, p) = & \sqrt{2} \sum_{c=1}^n p_c D_c^- - \sqrt{2} \sum_{c=1}^n \frac{\nu + \kappa \cosh(2q_c)}{\sinh(2q_c)} X_{2e_c}^{-, i} \\ & - 2\mu \sum_{1 \leq a < b \leq n} \left(\frac{X_{e_a - e_b}^{-, i}}{\sinh(q_a - q_b)} + \frac{X_{e_a + e_b}^{-, i}}{\sinh(q_a + q_b)} \right) - \kappa i C, \end{aligned} \quad (3.83)$$

from (3.82) it follows that B has the block matrix structure

$$B = \begin{bmatrix} S & T \\ T & S \end{bmatrix}, \quad (3.84)$$

where S and T are appropriate $\mathfrak{u}(n)$ -valued functions on \mathcal{P}^S . Namely, for their matrix entries we have

$$T_{c,c} = i \frac{\nu \cosh(2q_c) + \kappa}{\sinh^2(2q_c)}, \quad T_{a,b} = i\mu \frac{\cosh(q_a + q_b)}{\sinh^2(q_a + q_b)}, \quad S_{a,b} = -i\mu \frac{\cosh(q_a - q_b)}{\sinh^2(q_a - q_b)}, \quad (3.85)$$

meanwhile

$$S_{c,c} = i \frac{\nu + \kappa \cosh(2q_c)}{\sinh^2(2q_c)} + i\mu \sum_{\substack{d=1 \\ (d \neq c)}}^n \left(\frac{1}{\sinh^2(q_c - q_d)} + \frac{1}{\sinh^2(q_c + q_d)} \right), \quad (3.86)$$

where $a, b, c \in \mathbb{N}_n$ and $a \neq b$. Notice that B is actually a \mathfrak{k} -valued map depending only on q .

As we have discussed in Section 2, the reduced Hamiltonian corresponding to F_2 (2.37) coincides with the Hamiltonian of the Sutherland model (1.4), i.e. $H^S = \langle L, L \rangle / 4$. By applying the Hamiltonian vector field $\mathbf{X}_{H^S} \in \mathfrak{X}(\mathcal{P}^S)$ on L , from the r -matrix Poisson bracket (3.63) we obtain

$$\mathbf{X}_{H^S}[L] = \frac{1}{2}[RL, L] = [B, L]. \quad (3.87)$$

That is, along each trajectory $t \mapsto (q(t), p(t))$ of the Sutherland dynamics the Lax equation

$$\dot{L} = [B, L] \quad (3.88)$$

holds. The above observation can be sharpened as follows.

Theorem 5. *A smooth curve in the phase space \mathcal{P}^S (1.2) is an integral curve of the hyperbolic BC_n Sutherland dynamics, if and only if, along the curve the Lax equation (3.88) is satisfied.*

Proof. By applying repeatedly the identity

$$\frac{\cosh(x)}{\sinh^2(x)} \frac{1}{\sinh(y)} - \frac{\cosh(y)}{\sinh^2(y)} \frac{1}{\sinh(x)} = \frac{1}{\sinh(x+y)} \left(\frac{1}{\sinh^2(x)} - \frac{1}{\sinh^2(y)} \right), \quad (3.89)$$

elementary algebraic manipulations lead to the formula

$$\begin{aligned} [B, L] = & 2\mu \sum_{1 \leq a < b \leq n} \left((p_a - p_b) \frac{\cosh(q_a - q_b)}{\sinh^2(q_a - q_b)} X_{e_a - e_b}^{-,i} + (p_a + p_b) \frac{\cosh(q_a + q_b)}{\sinh^2(q_a + q_b)} X_{e_a + e_b}^{-,i} \right) \\ & + 2\sqrt{2} \sum_{c=1}^n p_c \frac{\nu \cosh(2q_c) + \kappa}{\sinh^2(2q_c)} X_{2e_c}^{-,i} - \sqrt{2} \sum_{c=1}^n \frac{\partial H}{\partial q_c} D_c^-. \end{aligned} \quad (3.90)$$

On the other hand, by differentiating L (3.83) along an arbitrary smooth curve $(q(t), p(t)) \in \mathcal{P}^S$ with respect to time t , we find easily that

$$\begin{aligned} \dot{L} = & 2\mu \sum_{1 \leq a < b \leq n} \left((\dot{q}_a - \dot{q}_b) \frac{\cosh(q_a - q_b)}{\sinh^2(q_a - q_b)} X_{e_a - e_b}^{-,i} + (\dot{q}_a + \dot{q}_b) \frac{\cosh(q_a + q_b)}{\sinh^2(q_a + q_b)} X_{e_a + e_b}^{-,i} \right) \\ & + 2\sqrt{2} \sum_{c=1}^n \dot{q}_c \frac{\nu \cosh(2q_c) + \kappa}{\sinh^2(2q_c)} X_{2e_c}^{-,i} + \sqrt{2} \sum_{c=1}^n \dot{p}_c D_c^-. \end{aligned} \quad (3.91)$$

Hence, by comparing formulae (3.90) and (3.91), we conclude that the Lax equation is equivalent to the Hamiltonian equation of motion of the Sutherland model. \square

4 Discussion

Starting with the seminal paper [15], a lot of effort has been devoted to explore the r -matrix structure of the Calogero–Moser–Sutherland many-particle systems. In this paper we contribute to this research area by constructing a dynamical r -matrix for the hyperbolic BC_n Sutherland model with three independent parameters (1.4). The outcome of our analysis is consistent with the results of [10] on the r -matrix structure of the C_n Sutherland model with two independent coupling constants. We wish to mention that the authors of paper [16] have also constructed a dynamical r -matrix for a restricted class of the BC_n -type Sutherland models. More precisely, their results are valid under the same restriction on the coupling parameters that was stuck to these models in the fundamental work of Olshanetsky and Perelomov [1]. Recall also that the BC_n -type r -matrix in [16] explicitly depends on the coupling parameters. Note, however, that our r -matrix (3.66) is *independent* of the coupling parameters, and so it is equally valid for the B_n , C_n and BC_n Sutherland models, too. This ‘universal’ feature of (3.66) naturally indicates a Yang–Baxter-type algebraic structure behind these models. We wish to investigate this important topic in future publications. A related open problem is to explore the relationship between our r -matrix and the BC_n -type Sutherland model with two types of particles (see e.g. [17], [18]).

Similar questions arise in the context of the elliptic Calogero models, too. We have a fairly complete understanding of the r -matrix structure of Krichever’s spectral parameter dependent Lax matrix [19] for the A_n -type model (see [20], [21]). These elliptic r -matrices are dynamical objects, depending on the particle coordinates. As is known, one can even construct a Lax matrix for the elliptic A_n -type model, which obeys an r -matrix Poisson bracket with Belavin’s [22] *non-dynamical* elliptic r -matrix.

(The details on the elliptic case can be found in [23], whereas [24] contains an elementary account on the non-dynamical r -matrix structure of the degenerate A_n -type models.) However, for the r -matrix structure of the elliptic BC_n Calogero model only partial results are known. Namely, in [16] a dynamical r -matrix is constructed for the elliptic BC_n Calogero model with the aforementioned restriction on coupling constants. For this restricted class of BC_n models the Lax representation with non-dynamical r -matrix has been also investigated (see [25]). Nevertheless, to our knowledge, the r -matrix of the most general three parameter dependent elliptic BC_n Calogero model is still missing. It also appears to be an interesting open problem to provide r -matrices for the universal Lax operators constructed in [7]. An equally ambitious project would be to construct an r -matrix for Inozemtsev's [26] many-parameter dependent elliptic model, too. We hope that appropriate generalizations of our r -matrix (3.66) may play a role in clarifying these issues.

To conclude the paper, let us recall that the Ruijsenaars–Schneider–van Diejen (RSvD) models (see e.g. [27], [28]) are natural generalizations of the Calogero–Moser–Sutherland (CMS) particle systems. The r -matrix structure of the A_n -type Ruijsenaars–Schneider models is well understood (for details on the elliptic models see e.g. [29], [30]), but for the generic non- A_n -type models even the Lax representation of the dynamics is missing. Quite surprisingly, the construction of a Lax matrix for the rational BC_n RSvD model with three independent coupling parameters was carried out only in the recent paper [9]. Due to the dual reduction picture presented in [9], we expect that the r -matrix structure of the rational BC_n RSvD model can be analyzed by the same techniques we outlined in Section 3. As for the A_n -type systems, it has been observed that in some sense the CMS and the RSvD models can be characterized by the *same* r -matrices (for details see [31]). It appears to be an interesting question whether the dual reduction picture behind the CMS and the RSvD models can provide a geometric explanation of this remarkable phenomenon. We wish to come back to these problems in future publications.

A Convenient basis for $\mathfrak{u}(n, n)$

As a supplementary material to the main text, in this appendix we present a convenient basis for the real Lie algebra $\mathfrak{g} = \mathfrak{u}(n, n)$ adapted to the orthogonal decomposition (2.10). First, for each $c \in \mathbb{N}_n$ we define the linear functional

$$e_c: \mathbb{R}^n \rightarrow \mathbb{R}, \quad q = (q_1, \dots, q_n) \mapsto e_c(q) = q_c. \quad (\text{A.1})$$

Clearly the set of functionals

$$\mathcal{R}_+ = \{e_a \pm e_b \mid 1 \leq a < b \leq n\} \cup \{2e_c \mid c \in \mathbb{N}_n\} \quad (\text{A.2})$$

can be seen as a family of *positive* roots of type C_n . We also need the standard $N \times N$ elementary matrices $e_{k,l}$. Recall that for their matrix entries we have $(e_{k,l})_{k',l'} = \delta_{k,k'}\delta_{l,l'}$.

Now, for each $c \in \mathbb{N}_n$ we define the diagonal matrices

$$D_c^+ = \frac{i}{\sqrt{2}}(e_{c,c} + e_{n+c,n+c}), \quad D_c^- = \frac{1}{\sqrt{2}}(e_{c,c} - e_{n+c,n+c}). \quad (\text{A.3})$$

Clearly $\{D_c^+\}$ is a basis in \mathfrak{m} , whereas $\{D_c^-\}$ is basis in \mathfrak{a} , satisfying the relations

$$\langle D_c^+, D_d^+ \rangle = -\delta_{c,d}, \quad \langle D_c^-, D_d^- \rangle = \delta_{c,d}. \quad (\text{A.4})$$

Next, for each $c \in \mathbb{N}_n$ we introduce the matrices

$$X_{2e_c}^{\pm, i} = -\frac{i}{\sqrt{2}}(e_{c, n+c} \pm e_{n+c, c}). \quad (\text{A.5})$$

Also, for all $1 \leq a < b \leq n$ we define the following matrices with purely real entries

$$\begin{aligned} X_{e_a - e_b}^{\pm, r} &= \frac{1}{2}(e_{a, b} \mp e_{b, a} \pm e_{n+a, n+b} - e_{n+b, n+a}), \\ X_{e_a + e_b}^{\pm, r} &= -\frac{1}{2}(e_{a, n+b} - e_{b, n+a} \pm e_{n+a, b} \mp e_{n+b, a}), \end{aligned} \quad (\text{A.6})$$

together with the following matrices with purely imaginary entries

$$\begin{aligned} X_{e_a - e_b}^{\pm, i} &= \frac{i}{2}(e_{a, b} \pm e_{b, a} \pm e_{n+a, n+b} + e_{n+b, n+a}), \\ X_{e_a + e_b}^{\pm, i} &= -\frac{i}{2}(e_{a, n+b} + e_{b, n+a} \pm e_{n+a, b} \pm e_{n+b, a}). \end{aligned} \quad (\text{A.7})$$

The set of vectors $\{X_\alpha^{+, \epsilon}\}$ forms a basis in \mathfrak{m}^\perp , meanwhile $\{X_\alpha^{-, \epsilon}\}$ is a basis in \mathfrak{a}^\perp . Note that

$$\langle X_\alpha^{+, \epsilon}, X_{\alpha'}^{+, \epsilon'} \rangle = -\delta_{\alpha, \alpha'} \delta_{\epsilon, \epsilon'}, \quad \langle X_\alpha^{-, \epsilon}, X_{\alpha'}^{-, \epsilon'} \rangle = \delta_{\alpha, \alpha'} \delta_{\epsilon, \epsilon'}. \quad (\text{A.8})$$

Due to the orthogonality relations (A.4) and (A.8), the construction of the corresponding dual basis is trivial. Keeping in mind the notation introduced in (2.7), it is worth mentioning that the above listed vectors satisfy the commutation relations

$$[Q, X_\alpha^{\pm, \epsilon}] = \alpha(q) X_\alpha^{\mp, \epsilon}, \quad (\text{A.9})$$

where $q \in \mathbb{R}^n$, $\alpha \in \mathcal{R}_+$ and $\epsilon \in \{r, i\}$.

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